1 Doppelpendel

Laut `http://en.wikipedia.org/wiki/Double_pendulum` ist das Doppelpendel mit Massen $m_1 = m_2 = m$ und Längen $l_1 = l_2 = l$ durch die 4 unabhängigen Variablen $\theta_1, \theta_2, p_{\theta_1}$ und p_{θ_2} spezifiert und es gilt

$$\begin{aligned}
p_{\theta_1} &= \frac{1}{6} m l^2 \left[8\dot\theta_1 + 3\dot\theta_2 \cos(\theta_1 - \theta_2) \right] \\
p_{\theta_2} &= \frac{1}{6} m l^2 \left[2\dot\theta_2 + 3\dot\theta_1 \cos(\theta_1 - \theta_2) \right] \\
\dot\theta_1 &= \frac{6}{ml^2} \cdot \frac{2p_{\theta_1} - 3\cos(\theta_1-\theta_2) p_{\theta_2}}{16 - 9\cos^2(\theta_1-\theta_2)} \\
\dot\theta_2 &= \frac{6}{ml^2} \cdot \frac{8p_{\theta_2} - 3\cos(\theta_1-\theta_2) p_{\theta_1}}{16 - 9\cos^2(\theta_1-\theta_2)} \\
\dot p_{\theta_1} &= -\frac{1}{2} m l^2 \left[\dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) + 3\frac{g}{l} \sin\theta_1 \right] \\
\dot p_{\theta_2} &= -\frac{1}{2} m l^2 \left[-\dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) + \frac{g}{l} \sin\theta_2 \right].
\end{aligned}$$

1.1 Flussfunktion

Ziel ist es, $\hat d$ kleinstmöglich zu halten, wobei $\hat d = \max_{(k,\vec\imath)} |\{j : \vec\imath_j \neq 0\}|$ ist, denn für $a_{\nu,n'}$ mit $n' \leqslant n$ und n so, dass $\frac{M\cdot R}{R-|z|}\left(\frac{|z|}{R}\right)^{n+1} \leqslant 2^p$, werden $\binom{\hat d + n' - 1}{n'} \in \mathcal{O}^*(n^{\hat d - 1})$ Additionen von Termen $c_{\nu,k,\vec\imath} \prod_{j:\vec\imath_j \neq 0} a_{\nu,n_j}^{(\vec\imath_j)}$ durchgeführt.

Folgender 11-dimensionaler Fluss erreicht $\hat d = 4$ und führt damit zu kubischer Laufzeit in n (und damit grob auch in $\mathcal{M}(p)$):

$$F : (t, (z_1, \ldots, z_{11})) \mapsto (\dot z_1, \ldots, \dot z_{11}), \text{ wobei}$$

$$\begin{aligned}
z_1 &= \theta_1 \\
z_2 &= \theta_2 \\
z_3 &= p_{\theta_1} \\
z_4 &= p_{\theta_2} \\
z_5 &= \sin\theta_1 \\
z_6 &= \sin\theta_2 \\
z_7 &= \cos\theta_1 \\
z_8 &= \cos\theta_2 \\
z_9 &= \left[16 - 9\cos^2(\theta_1 - \theta_2)\right]^{-1} \\
z_{10} &= \sin(\theta_1 - \theta_2) \\
z_{11} &= \cos(\theta_1 - \theta_2)
\end{aligned}$$

Unter Verwendung der Identität $z_{11}^2 = \cos^2(\theta_1 - \theta_2) = 1 - \sin^2(\theta_1 - \theta_2) = 1 - z_{10}^2$ ergeben sich

$$
\begin{aligned}
\dot{z}_1 &= \dot{\theta}_1 = \frac{6}{ml^2} \cdot \frac{2p_{\theta_1} - 3\cos(\theta_1 - \theta_2)p_{\theta_2}}{16 - 9\cos^2(\theta_1 - \theta_2)} = \frac{6}{ml^2} \cdot (2z_3 - 3z_{11}z_4)z_9 = \frac{12}{ml^2}z_3z_9 - \frac{18}{ml^2}z_4z_9z_{11} \\
\dot{z}_2 &= \dot{\theta}_2 = \frac{6}{ml^2} \cdot \frac{8p_{\theta_2} - 3\cos(\theta_1 - \theta_2)p_{\theta_1}}{16 - 9\cos^2(\theta_1 - \theta_2)} = \frac{6}{ml^2} \cdot (8z_4 - 3z_{11}z_3)z_9 = \frac{48}{ml^2}z_4z_9 - \frac{18}{ml^2}z_3z_9z_{11} \\
\dot{z}_3 &= \dot{p}_{\theta_1} = -\frac{1}{2}ml^2\left[\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) + 3\frac{g}{l}\sin\theta_1\right] \\
&= -\frac{1}{2}ml^2\left[\frac{6}{ml^2} \cdot (2z_3 - 3z_{11}z_4)z_9 \cdot \frac{6}{ml^2} \cdot (8z_4 - 3z_{11}z_3)z_9 \cdot z_{10} + 3\frac{g}{l}z_5\right] \\
&= -\frac{6}{2} \cdot \frac{6}{ml^2}z_9^2z_{10}(16z_3z_4 - 6z_3^2z_{11} - 24z_4^2z_{11} + 9z_3z_4z_{11}^2) - \frac{3}{2}ml^2 \cdot \frac{g}{l}z_5 \\
&= -\frac{18}{ml^2}(16z_3z_4z_9^2z_{10} - 6z_3^2z_9^2z_{10}z_{11} - 24z_4^2z_9^2z_{10}z_{11} + 9z_3z_4z_9^2z_{10}(1 - z_{10}^2)) - \frac{3}{2}mlgz_5 \\
&= -\frac{450}{ml^2}z_3z_4z_9^2z_{10} + \frac{108}{ml^2}z_3^2z_9^2z_{10}z_{11} + \frac{432}{ml^2}z_4^2z_9^2z_{10}z_{11} + \frac{162}{ml^2}z_3z_4z_9^2z_{10}^3 - \frac{3}{2}mlgz_5 \\
\dot{z}_4 &= \dot{p}_{\theta_2} = -\frac{1}{2}ml^2\left[-\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) + \frac{g}{l}\sin\theta_2\right] \\
&= \frac{450}{ml^2}z_3z_4z_9^2z_{10} - \frac{108}{ml^2}z_3^2z_9^2z_{10}z_{11} - \frac{432}{ml^2}z_4^2z_9^2z_{10}z_{11} - \frac{162}{ml^2}z_3z_4z_9^2z_{10}^3 - \frac{1}{2}mlgz_6 \\
\dot{z}_5 &= (\sin\theta_1)' = \dot{\theta}_1\cos\theta_1 = \frac{6}{ml^2} \cdot (2z_3 - 3z_{11}z_4)z_9z_7 = \frac{12}{ml^2}z_3z_7z_9 - \frac{18}{ml^2}z_4z_7z_9z_{11} \\
\dot{z}_6 &= (\sin\theta_2)' = \dot{\theta}_2\cos\theta_2 = \frac{6}{ml^2} \cdot (8z_4 - 3z_{11}z_3)z_9z_8 = \frac{48}{ml^2}z_4z_8z_9 - \frac{18}{ml^2}z_3z_8z_9z_{11} \\
\dot{z}_7 &= (\cos\theta_1)' = -\dot{\theta}_1\sin\theta_1 = -\frac{6}{ml^2} \cdot (2z_3 - 3z_{11}z_4)z_9z_5 = -\frac{12}{ml^2}z_3z_5z_9 + \frac{18}{ml^2}z_4z_5z_9z_{11} \\
\dot{z}_8 &= (\cos\theta_2)' = -\dot{\theta}_2\sin\theta_2 = -\frac{6}{ml^2} \cdot (8z_4 - 3z_{11}z_3)z_9z_6 = -\frac{48}{ml^2}z_4z_6z_9 + \frac{18}{ml^2}z_3z_6z_9z_{11} \\
\dot{z}_9 &= \left(\left[16 - 9\cos^2(\theta_1 - \theta_2)\right]^{-1}\right)' \\
&= (-1) \cdot \left[16 - 9\cos^2(\theta_1 - \theta_2)\right]^{-2} \cdot \left[-9 \cdot 2\cos(\theta_1 - \theta_2) \cdot \left(-\sin(\theta_1 - \theta_2) \cdot [\dot{\theta}_1 - \dot{\theta}_2]\right)\right] \\
&= -18z_9^2z_{11}z_{10}\left[\left(\frac{6}{ml^2} \cdot (2z_3 - 3z_{11}z_4)z_9\right) - \left(\frac{6}{ml^2} \cdot (8z_4 - 3z_{11}z_3)z_9\right)\right] \\
&= -18z_9^2z_{11}z_{10} \cdot \frac{6}{ml^2}z_9\left[(2z_3 - 3z_{11}z_4) - (8z_4 - 3z_{11}z_3)\right] \\
&= -\frac{108}{ml^2}z_9^3z_{10}z_{11}(2z_3 - 3z_{11}z_4 - 8z_4 + 3z_{11}z_3) \\
&= -\frac{216}{ml^2}z_3z_9^3z_{10}z_{11} + \frac{324}{ml^2}z_4z_9^3z_{10}z_{11}^2 + \frac{864}{ml^2}z_4z_9^3z_{10}z_{11} - \frac{324}{ml^2}z_3z_9^3z_{10}z_{11}^2 \\
\dot{z}_{10} &= [\sin(\theta_1 - \theta_2)]' = \cos(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \\
&= \frac{6}{ml^2}z_9z_{11}\left[(2z_3 - 3z_{11}z_4) - (8z_4 - 3z_{11}z_3)\right] \\
&= \frac{12}{ml^2}z_3z_9z_{11} - \frac{18}{ml^2}z_4z_9z_{11}^2 - \frac{48}{ml^2}z_4z_9z_{11} + \frac{18}{ml^2}z_3z_9z_{11}^2 \\
\dot{z}_{11} &= [\cos(\theta_1 - \theta_2)]' = -\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \\
&= -\frac{6}{ml^2}z_9z_{10}\left[(2z_3 - 3z_{11}z_4) - (8z_4 - 3z_{11}z_3)\right] \\
&= -\frac{12}{ml^2}z_3z_9z_{10} + \frac{18}{ml^2}z_4z_9z_{10}z_{11} + \frac{48}{ml^2}z_4z_9z_{10} - \frac{18}{ml^2}z_3z_9z_{10}z_{11}
\end{aligned}
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